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Research Article

Positive Solutions to Singular and Delay Higher-Order Differential Equations on Time Scales

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We are concerned with singular three-point boundary value problems for delay higher-order dynamic equations on time scales. Theorems on the existence of positive solutions are obtained by utilizing the fixed point theorem of cone expansion and compression type. An example is given to illustrate our main result.

1. Introduction

In this paper, we are concerned with the following singular three-point boundary value problem (BVP for short) for delay higher-order dynamic equations on time scales:

$$
\begin{gathered}
(-1)^{n} u^{\Delta^{2n}}(t) = w(t) f(t, u(t-c)), \quad t \in [a,b], \\
u(t) = \psi(t), \quad t \in [a-c, a), \\
u^{\Delta^{2i}}(a) - \beta_{i+1} u^{\Delta^{2i+1}}(a) = \alpha_{i+1} u^{\Delta^{2i}}(\varpi), \\
\gamma_{i+1} u^{\Delta^{2i}}(\varpi) = u^{\Delta^{2i}}(b), \quad 0 \leq i \leq n-1,
\end{gathered}
\tag{1.1}
$$

where $c \in [0,(b-a)/2]$, $\varpi \in (a,b)$, $\beta_i \geq 0$, $1 < \gamma_i < (b-a+\beta_i)/(\varpi - a + \beta_i)$, $0 \leq \alpha_i < (b - \gamma_i \varpi + (\gamma_i - 1)(a - \beta_i))/(b - \varpi)$, $i = 1, 2, \ldots, n$ and $\psi \in C([a-c, a])$. The functional $w : (a,b) \to [0,+\infty)$ is continuous and $f : [a,b] \times (0,+\infty) \to [0,+\infty)$ is continuous. Our nonlinearity w may have singularity at $t = a$ and/or $t = b$, and f may have singularity at $u = 0$.

To understand the notations used in (1.1), we recall the following definitions which can be found in [1, 2].

(a) A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$,

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \qquad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \tag{1.2}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) < t$, respectively. If $\mathbb{T}$ has a left-scattered maximum t_1 (right-scattered minimum t_2), define $\mathbb{T}^k = \mathbb{T} - \{t_1\}$ ($\mathbb{T}_k = \mathbb{T} - \{t_2\}$); otherwise, set $\mathbb{T}^k = \mathbb{T}$ ($\mathbb{T}_k = \mathbb{T}$). By an interval $[a, b]$ we always mean the intersection of the real interval $[a, b]$ with the given time scale, that is, $[a, b] \cap \mathbb{T}$. Other types of intervals are defined similarly.

(b) For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the Δ-derivative of f at t, denoted by $f^{\Delta}(t)$, is the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U. \tag{1.3}$$

(c) For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the ∇-derivative of f at t, denoted by $f^{\nabla}(t)$, is the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)\left[\rho(t) - s\right] \right| \leq \varepsilon \left|\rho(t) - s\right|, \quad \forall s \in U. \tag{1.4}$$

(d) If $F^{\Delta}(t) = f(t)$ $(\Phi^{\nabla}(t) = g(t))$, then we define the integral

$$\int_a^t f(\varpi)\Delta\varpi = F(t) - F(a) \qquad \left(\int_a^t g(\varpi)\nabla\varpi = \Phi(t) - \Phi(a)\right). \tag{1.5}$$

Theoretically, dynamic equations on time scales can build bridges between continuous and discrete mathematics. Practically, dynamic equations have been proposed as models in the study of insect population models, neural networks, and many physical phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, chemically reacting systems as well as concentration in chemical of biological problems [2]. Hence, two-point and multipoint boundary value problems for dynamic equations on time scales have attracted many researchers' attention (see, e.g., [1–19] and references therein). Moreover, singular boundary value problems have also been treated in many papers (see, e.g., [4, 5, 12–14, 18] and references therein).

In 2004, J. J. DaCunha et al. [13] considered singular second-order three-point boundary value problems on time scales

$$u^{\Delta\Delta}(t) + f(t, u(t)) = 0, \quad (0,1] \cap \mathbb{T},$$
$$u(0) = 0, \quad u(p) = u\left(\sigma^2(1)\right) \tag{1.6}$$

and obtained the existence of positive solutions by using a fixed point theorem due to Gatica et al. [14], where $f : (0,1] \times (0,\infty) \to (0,\infty)$ is decreasing in u for every $t \in (0,1]$ and may have singularity at $u = 0$.

In 2006, Boey and Wong [11] were concerned with higher-order differential equation on time scales of the form

$$(-1)^{n-1} y^{\Delta^n}(t) = (-1)^{p+1} F\left(t, y\left(\sigma^{n-1}(t)\right)\right), \quad t \in [a,b],$$
$$y^{\Delta^i}(a) = 0, \quad 0 \le i \le p-1, \tag{1.7}$$
$$y^{\Delta^i}(\sigma(b)) = 0, \quad p \le i \le n-1,$$

where p, n are fixed integers satisfying $n \ge 2$, $1 \le p \le n-1$. They obtained some existence theorems of positive solutions by using Krasnosel'skii fixed point theorem.

Recently, Anderson and Karaca [8] studied higher-order three-point boundary value problems on time scales and obtained criteria for the existence of positive solutions.

The purpose of this paper is to investigate further the singular BVP for delay higher-order dynamic equation (1.1). By the use of the fixed point theorem of cone expansion and compression type, results on the existence of positive solutions to the BVP (1.1) are established.

The paper is organized as follows. In Section 2, we give some lemmas, which will be required in the proof of our main theorem. In Section 3, we prove some theorems on the existence of positive solutions for BVP (1.1). Moreover, we give an example to illustrate our main result.

2. Lemmas

For $1 \le i \le n$, let $G_i(t,s)$ be Green's function of the following three-point boundary value problem:

$$-u^{\Delta\Delta}(t) = 0, \quad t \in [a,b],$$
$$u(a) - \beta_i u^{\Delta}(a) = \alpha_i u(\varpi), \qquad \gamma_i u(\varpi) = u(b), \tag{2.1}$$

where $\varpi \in (a,b)$ and $\alpha_i, \beta_i, \gamma_i$ satisfy the following condition:

(C)

$$\beta_i \ge 0, \quad 1 < \gamma_i < \frac{b - a + \beta_i}{\varpi - a + \beta_i}, \qquad 0 \le \alpha_i < \frac{b - \gamma_i \varpi + (\gamma_i - 1)(a - \beta_i)}{b - \varpi}. \tag{2.2}$$

Throughout the paper, we assume that $\sigma(b) = b$.

From [8], we know that for any $(t,s) \in [a,b] \times [a,b]$ and $1 \le i \le n$,

$$G_i(t,s) = \begin{cases} G_{i_1}(t,s), & s \in [a,\varpi], \\ G_{i_2}(t,s), & s \in [\varpi,b], \end{cases} \tag{2.3}$$

where

$$\begin{gathered} G_{i_1}(t,s) = \frac{1}{d_i}\begin{cases} \left[\gamma_i(t-\varpi)+b-t\right]\left(\sigma(s)+\beta_i-a\right), & \sigma(s) \le t, \\ \left[\gamma_i(\sigma(s)-\overline{\omega})+b-\sigma(s)\right]\left(t+\beta_i-a\right)+\alpha_i(\varpi-b)(t-\sigma(s)), & t \le s, \end{cases} \\ G_{i_2}(t,s) = \frac{1}{d_i}\begin{cases} \left[\sigma(s)(1-\alpha_i)+\alpha_i\varpi+\beta_i-a\right](b-t)+\gamma_i\left(\varpi-a+\beta_i\right)(t-\sigma(s)), & \sigma(s) \le t, \\ \left[t(1-\alpha_i)+\alpha_i\varpi+\beta_i-a\right](b-\sigma(s)), & t \le s, \end{cases} \\ d_i = \left(\gamma_i-1\right)\left(a-\beta_i\right)+(1-\alpha_i)b+\varpi\left(\alpha_i-\gamma_i\right). \end{gathered} \tag{2.4}$$

The following four lemmas can be found in [8].

Lemma 2.1. *Suppose that the condition (C) holds. Then the Green function of* $G_i(t,s)$ *in* (2.3) *satisfies*

$$G_i(t,s) > 0, \quad (t,s) \in (a,b) \times (a,b). \tag{2.5}$$

Lemma 2.2. *Assume that the condition (C) holds. Then Green's function* $G_i(t,s)$ *in* (2.3) *satisfies*

$$G_i(t,s) \le \max\{G_i(b,s), G_i(\sigma(s),s)\}, \quad (t,s) \in [a,b] \times [a,b]. \tag{2.6}$$

Remark 2.3. (1) If $s \in ((\gamma_i(\varpi-a+\beta_i)-\alpha_i\varpi-\beta_i+a)/(1-\alpha_i), b], s \le t$, we know that $G_i(t,s)$ is nonincreasing in t and

$$\begin{aligned} \frac{G_i(b,s)}{G_i(\sigma(s),s)} &= \frac{\gamma_i\left(\varpi-a+\beta_i\right)(b-\sigma(s))}{\left(\sigma(s)(1-\alpha_i)+\alpha_i\varpi+\beta_i-a\right)(b-\sigma(s))} \\ &\ge \frac{\gamma_i\left(\varpi-a+\beta_i\right)}{b(1-\alpha_i)+\alpha_i\varpi+\beta_i-a} > 0. \end{aligned} \tag{2.7}$$

Therefore, we have

$$G_i(b,s) \le G_i(t,s) \le G_i(\sigma(s),s) \le \delta_i G_i(b,s), \tag{2.8}$$

where

$$\delta_i = \frac{b(1-\alpha_i)+\alpha_i\varpi+\beta_i-a}{\gamma_i\left(\varpi-a+\beta_i\right)} > 1. \tag{2.9}$$

(2) If t and s satisfy the other cases, then we get that $G_i(t,s)$ is nondecreasing in t and

$$G_i(t,s) \le G_i(b,s). \tag{2.10}$$

Lemma 2.4. *Assume that (C) holds. Then Green's function $G_i(t,s)$ in* (2.3) *verifies the following inequality:*

$$\begin{aligned} G_i(t,s) &\ge \min\left\{\frac{t-a}{b-a}, \frac{b-t}{\gamma_i(b-a)}\right\} G_i(b,s) \\ &\ge \min\left\{\frac{t-a}{\delta_i(b-a)}, \frac{b-t}{\gamma_i(b-a)}\right\} \max\{G_i(b,s), G_i(\sigma(s),s)\}. \end{aligned} \tag{2.11}$$

Remark 2.5. If $s \in [\varpi, (\gamma_i(\varpi - a + \beta_i) - \alpha_i\varpi - \beta_i + a)/(1-\alpha_i)), s \le t$, then we find

$$(\gamma_i - 1)(a - \beta_i) + (1-\alpha_i)\sigma(s) + \varpi(\alpha_i - \gamma_i) < 0. \tag{2.12}$$

So there exists a misprint on [8, Page 2431, line 23]. From (2.3), it follows that

$$\begin{aligned} \frac{G_i(t,s)}{G_i(b,s)} &= \frac{[\sigma(s)(1-\alpha_i) + \alpha_i\varpi + \beta_i - a](b-t) + \gamma_i(\varpi - a + \beta_i)(t - \sigma(s))}{\gamma_i(\varpi - a + \beta_i)(b - \sigma(s))} \\ &\ge \frac{(\varpi + \beta_i - a)(b-t) + \gamma_i(\varpi - a + \beta_i)(t - \sigma(s))}{\gamma_i(\varpi - a + \beta_i)(b-a)} \ge \frac{b-t}{\gamma_i(b-a)}. \end{aligned} \tag{2.13}$$

Consequently, we get

$$G_i(t,s) \ge \frac{b-t}{\gamma_i(b-a)} G_i(b,s). \tag{2.14}$$

If $s \in ((\gamma_i(\varpi - a + \beta_i) - \alpha_i\varpi - \beta_i + a)/(1-\alpha_i), b]$, $s \le t$, then, from (2.8), we obtain

$$G_i(t,s) \ge \frac{t-a}{b-a} G_i(b,s) \ge \frac{t-a}{\delta_i(b-a)} G_i(\sigma(s),s). \tag{2.15}$$

Remark 2.6. If we set $h_i(t) := \min\{(t-a)/\delta_i(b-a), (b-t)/\gamma_i(b-a)\}$, then we have

$$G_i(t,s) \geq h_i(t)\ \max\{G_i(b,s), G_i(\sigma(s),s)\}, \quad (t,s) \in [a,b] \times [a,b]. \tag{2.16}$$

Denote

$$\|G_i(\cdot,s)\| = \max_{t\in[a,b]} |G_i(t,s)|, \quad s \in [a,b]. \tag{2.17}$$

Thus we have

$$G_i(t,s) \geq h_i(t)\|G_i(\cdot,s)\|, \quad (t,s) \in [a,b] \times [a,b]. \tag{2.18}$$

Lemma 2.7. *Assume that condition (C) is satisfied. For $G_i(t,s)$ as in* (2.3)*, put $H_1(t,s) := G_1(t,s)$ and recursively define*

$$H_j(t,s) = \int_a^b H_{j-1}(t,r) G_j(r,s)\Delta r \tag{2.19}$$

for $2 \leq j \leq n$. Then $H_n(t,s)$ is Green's function for the homogeneous problem

$$\begin{gathered}(-1)^n u^{\Delta^{2n}}(t) = 0, \quad t \in [a,b], \\ u^{\Delta^{2i}}(a) - \beta_{i+1} u^{\Delta^{2i+1}}(a) = \alpha_{i+1} u^{\Delta^{2i}}(\varpi), \\ \gamma_{i+1} u^{\Delta^{2i}}(\varpi) = u^{\Delta^{2i}}(b), \quad 0 \leq i \leq n-1.\end{gathered} \tag{2.20}$$

Lemma 2.8. *Assume that (C) holds. Denote*

$$K := \prod_{j=1}^{n-1} k_j, \qquad L := \prod_{j=1}^{n-1} l_j, \tag{2.21}$$

then Green's function $H_n(t,s)$ in Lemma 2.7 satisfies

$$h_1(t) L \|G_n(\cdot,s)\| \leq H_n(t,s) \leq K\|G_n(\cdot,s)\|, \quad (t,s) \in [a,b] \times [a,b], \tag{2.22}$$

where

$$k_j = \int_a^b \|G_j(\cdot,s)\|\Delta s > 0, \quad l_j = \int_a^b \|G_j(\cdot,s)\| h_{j+1}(s)\Delta s, \quad 1 \leq j \leq n-1. \tag{2.23}$$

Proof. We proceed by induction on $n \geq 2$. We denote the statement by $P(n)$. From Lemma 2.7, it follows that

$$\begin{aligned}\|H_2(t,s)\| &= \left\| \int_a^b H_1(t,r) G_2(r,s) \Delta r \right\| \\ &\leq \int_a^b \|G_1(\cdot,r)\| \|G_2(\cdot,s)\| \Delta r = k_1 \|G_2(\cdot,s)\|,\end{aligned} \tag{2.24}$$

and from (2.18), we have

$$\begin{aligned}H_2(t,s) &= \int_a^b H_1(t,r) G_2(r,s) \Delta r \\ &\geq \int_a^b h_1(t) \|G_1(\cdot,r)\| \times h_2(r) \|G_2(\cdot,s)\| \Delta r \\ &= h_1(t) l_1 \|G_2(\cdot,s)\|.\end{aligned} \tag{2.25}$$

So $P(2)$ is true. □

We now assume that $P(m)$ is true for some positive integer $m \geq 2$. From Lemma 2.7, it follows that

$$\begin{aligned}\|H_{m+1}(t,s)\| &= \left\| \int_a^b H_m(t,r) G_{m+1}(r,s) \Delta r \right\| \\ &\leq \int_a^b H_m(t,r) \|G_{m+1}(r,s)\| \Delta r \\ &\leq \left(\int_a^b \prod_{j=1}^{m-1} k_j \times \|G_m(\cdot,r)\| \Delta r \right) \|G_{m+1}(\cdot,s)\| \\ &= \prod_{j=1}^{m} k_j \|G_{m+1}(\cdot,s)\|, \\ H_{m+1}(t,s) &= \int_a^b H_m(t,r) G_{m+1}(r,s) \Delta r \\ &\geq \int_a^b h_1(t) \times \prod_{j=1}^{m-1} l_j G_m(\cdot,r) h_{m+1}(r) \|G_{m+1}(\cdot,s)\| \Delta r \\ &= h_1(t) \prod_{j=1}^{m} l_j \|G_{m+1}(\cdot,s)\|.\end{aligned} \tag{2.26}$$

So $P(m+1)$ holds. Thus $P(n)$ is true by induction.

Lemma 2.9 (see [20]). *Let $(E, \|\cdot\|)$ be a real Banach space and $P \subset E$ a cone. Assume that $T : P_{\zeta,\eta} \to P$ is completely continuous operator such that*

(i) $Tu \nleq u$ *for* $u \in \partial P_\zeta$ *and* $Tu \ngeq u$ *for* $u \in \partial P_\eta$,

(ii) $Tu \ngeq u$ *for* $u \in \partial P_\zeta$ *and* $Tu \nleq u$ *for* $u \in \partial P_\eta$.

Then T has a fixed point $u^* \in P$ with $\zeta \leq \|u^*\| \leq \eta$.

3. Main Results

We assume that $\{a_m\}_{m\geq 1}$ and $\{b_m\}_{m\geq 1}$ are strictly decreasing and strictly increasing sequences, respectively, with $\lim_{m\to\infty} a_m = a$, $\lim_{m\to\infty} b_m = b$ and $a_1 < b_1$. A Banach space $E = C([a,b])$ is the set of real-valued continuous (in the topology of $\mathbb{T}$) functions $u(t)$ defined on $[a,b]$ with the norm

$$\|u\| = \max_{t\in[a,b]} |u(t)|. \tag{3.1}$$

Define a cone by

$$P = \left\{ u \in E : u(t) \geq \frac{h_1(t)L}{K}\|u\|,\ t \in [a,b] \right\}. \tag{3.2}$$

Set

$$\begin{gathered} P_\xi = \{u \in P : \|u\| < \xi\}, \quad \partial P_\xi = \{u \in P : \|u\| = \xi\}, \quad \xi > 0, \\ P_{\zeta,\eta} = \{u \in P : \zeta < \|u\| < \eta\}, \quad 0 < \zeta < \eta, \\ Y_1 = \{t \in [a,b] : t - c < a\}, \qquad Y_2 = \{t \in [a,b] : t - c \geq a\}, \\ Y_m = \{t \in Y_2 : t - c \in [a, a_m] \cup [b_m, b]\}. \end{gathered} \tag{3.3}$$

Assume that

(C1) $\varphi : [a-c, a] \to (0,\infty)$ is continuous;

(C2) we have

$$0 < K\int_p^q \|G_n(\cdot,s)\| w(s)\Delta s, \qquad K\int_a^b \|G_n(\cdot,s)\| w(s)\Delta s < +\infty, \tag{3.4}$$

for constants p and q with $a + c < p < q < b$;

(C3) the function $f : [a,b] \times (0,+\infty) \to \mathbb{R}^+$ is continuous and $w : (a,b) \to \mathbb{R}^+$ is continuous satisfying

$$\lim_{m\to\infty} \sup_{u\in P_{\zeta,\eta}} K\int_{Y_m} \|G_n(\cdot,s)\| w(s) f(s, u(s-c))\Delta s = 0, \quad \forall 0 < \zeta < \eta. \tag{3.5}$$

We seek positive solutions $u : [a,b] \to \mathbb{R}^+$, satisfying (1.1). For this end, we transform (1.1) into an integral equation involving the appropriate Green function and seek fixed points of the following integral operator.

Define an operator $T : C^+[a,b] \to C[a,b]$ by

$$(Tu)(t) = \int_a^b H_n(t,s)w(s)f(s,u(s-c))\Delta s, \quad \forall u \in C^+([a,b]), \tag{3.6}$$

where $C^+[a,b] = \{u \in C[a,b] \mid \ u(t) \geq 0,\ t \in [a,b]\}$.

Proposition 3.1. *Let (C1), (C2), and (C3) hold, and let ζ, η be fixed constants with $0 < \zeta < \eta$. Then $T : P_{\zeta,\eta} \to P$ is completely continuous.*

Proof. We separate the proof into four steps.

Step 1. For each $u \in P_{\zeta,\eta}$, Tu is bounded.

By condition (C3), there exists some positive integer m_0 satisfying

$$\sup_{u\in P_{\zeta,\eta}} K\int_{\Upsilon_{m_0}} \|G_n(\cdot,s)\|w(s)f(s,u(s-c))\Delta s \leq 1, \tag{3.7}$$

where

$$\Upsilon_{m_0} = \{t \in \Upsilon_2 : t-c \in [a,a_{m_0}] \cup [b_{m_0},b]\}; \tag{3.8}$$

here, we used the fact that for each $u \in P_{\zeta,\eta}$ and $t \in [a_{m_0}+c, b_{m_0}+c] \cap [a,b]$,

$$\eta \geq u(t-c) \geq \frac{h_1(t-c)L}{K}\|u\| \geq \zeta \min\left\{\frac{h_1(a_{m_0})L}{K}, \frac{h_1(b_{m_0})L}{K}, \frac{h_1(b-c)L}{K}\right\} = \zeta h > 0, \tag{3.9}$$

where

$$h = \min\left\{\frac{h_1(a_{m_0})L}{K}, \frac{h_1(b_{m_0})L}{K}, \frac{h_1(b-c)L}{K}\right\}. \tag{3.10}$$

Set

$$\begin{aligned} D &:= \max\{f(t,\varphi(t-c)) : t \in \Upsilon_1\}, \\ Q &:= \max\{f(t,u(t-c)) : t \in \Upsilon_2, \zeta h \leq u(t-c) \leq \eta\}. \end{aligned} \tag{3.11}$$

Then we obtain

$$
\begin{aligned}
Tu(t) &\le \sup_{t\in[a,b]} \sup_{u\in P_{\zeta,\eta}} \int_a^b H_n(t,s)w(s)f(s,u(s-c))\Delta s \\
&\le K \sup_{u\in P_{\zeta,\eta}} \int_{Y_1} \|G_n(\cdot,s)\|w(s)f(s,u(s-c))\Delta s \\
&\quad + \sup_{u\in P_{\zeta,\eta}} K \int_{Y_{m_0}} \|G_n(\cdot,s)\|w(s)f(s,u(s-c))\Delta s \\
&\quad + \sup_{u\in P_{\zeta,\eta}} K \int_{Y_2\setminus Y_{m_0}} \|G_n(\cdot,s)\|w(s)f(s,u(s-c))\Delta s \\
&\le 1 + \max\{D,Q\} K \int_a^b \|G_n(\cdot,s)\|w(s)\Delta s < +\infty.
\end{aligned}
\tag{3.12}
$$

Consequently, Tu is bounded and well defined.

Step 2. $T : P_{\zeta,\eta} \to P$. For every $u \in P_{\zeta,\eta}$, we get from (2.22)

$$
\begin{aligned}
\|Tu\| &= \sup_{t\in[a,b]} \int_a^b H_n(t,s)w(s)f(s,u(s-c))\Delta s \\
&\le K \int_a^b \|G_n(\cdot,s)\|w(s)f(s,u(s-c))\Delta s.
\end{aligned}
\tag{3.13}
$$

Then by the above inequality

$$
\begin{aligned}
(Tu)(t) &= \int_a^b H_n(t,s)w(s)f(s,u(s-c))\Delta s \\
&\ge \int_a^b h_1(t)L\|G_n(\cdot,s)\|w(s)f(s,u(s-c))\Delta s \\
&\ge \frac{h_1(t)L}{K}\|Tu\|.
\end{aligned}
\tag{3.14}
$$

This leads to $Tu \in P$.

Step 3. We will show that $T : P_{\zeta,\eta} \to P$ is continuous. Let $\{u_m\}_{m\geq 1}$ be any sequence in $P_{\zeta,\eta}$ such that $\lim_{m\to\infty} u_m = u \in P_{\zeta,\eta}$. Notice also that as $m \to \infty$,

$$\begin{aligned}
&\phi_m(s) = \left| f(s, u_m(s-c)) - f(s, u(s-c)) \right| w(s) \longrightarrow 0, \quad \text{for } s \in (a+c, b),\\
&\left| f(s, u_m(s-c)) - f(s, u(s-c)) \right| w(s)\\
&\quad = \left| f(s, \psi(s-c)) - f(s, \psi(s-c)) \right| w(s) = 0, \quad \text{for } s \in (a, a+c),\\
&\int_{Y_2} H_n(t,s)\phi_m(s)\Delta s \leq \sup_{x\in P_{\zeta,\eta}} 2K \int_{Y_2} \|G_n(\cdot, s)\| w(s) f(s, x(s))\Delta s < +\infty.
\end{aligned} \tag{3.15}$$

Now these together with (C2) and the Lebesgue dominated convergence theorem [10] yield that as $m \to \infty$,

$$\|Tu_m - Tu\| = \sup_{t\in[a,b]} \int_a^b H_n(t,s) w(s) \left| f(s, u_m(s-c)) - f(s, u(s-c)) \right| \Delta s \longrightarrow 0. \tag{3.16}$$

Step 4. $T : P_{\zeta,\eta} \to P$ is compact.

Define

$$\begin{aligned}
&w_m(t) = \begin{cases} \min\{w(t), w(a_m)\}, & a \leq t \leq a_m,\\ w(t), & a_m \leq t \leq b_m,\\ \min\{w(t), w(b_m)\}, & b_m \leq t \leq b, \end{cases}\\
&f_m(t, u(t-c)) = \begin{cases} f(t, \psi(t-c)), & a \leq t < a+c,\\ \min\{f(t, u(t-c)), f(t, u(a_m))\}, & a+c \leq t \leq a_m + c,\\ f(t, u(t-c)), & t \in [a_m + c, b_m + c] \cap [a,b],\\ \min\{f(t, u(t-c)), f(t, u(b_m))\}, & t \in [b_m + c, b] \cap [a,b], \end{cases}
\end{aligned} \tag{3.17}$$

and an operator sequence $\{T_m\}$ for a fixed m by

$$(T_m u)(t) = \int_a^b H_n(t,s) w_m(s) f_m(s, u(s-c))\Delta s, \quad \forall t \in [a,b]. \tag{3.18}$$

Clearly, the operator sequence $\{T_m\}$ is compact by using the Arzela-Ascoli theorem [3], for each $m \in \mathbb{N}$. We will prove that T_m converges uniformly to T on $P_{\zeta,\eta}$. For any $u \in P_{\zeta,\eta}$, we obtain

$$\begin{aligned}\|T_m u - Tu\| &= \sup_{t\in[a,b]}\left|\int_a^b H_n(t,s)\big(w_m(s)f_m(s,u(s-c)) - w(s)f(s,u(s-c))\big)\Delta s\right| \\ &\le K\int_a^b \|G_n(\cdot,s)\|\big|w_m(s)f_m(s,u(s-c)) - w(s)f(s,u(s-c))\big|\Delta s \\ &\le K\int_{Y_1}\|G_n(\cdot,s)\||w_m(s)-w(s)|f\big(s,\varphi(s-c)\big)\Delta s \\ &\quad + K\int_{Y_2}\|G_n(\cdot,s)\|\big|w_m(s)f_m(s,u(s-c)) - w(s)f(s,u(s-c))\big|\Delta s.\end{aligned} \tag{3.19}$$

From (C1), (C2), and the Lebesgue dominated convergence theorem [10], we see that the right-hand side (3.19) can be sufficiently small for $m$being big enough. Hence the sequence $\{T_m\}$ of compact operators converges uniformly to T on $P_{\zeta,\eta}$ so that operator T is compact. Consequently, $T : P_{\zeta,\eta} \to P$ is completely continuous by using the Arzela-Ascoli theorem [3]. □

Proposition 3.2. *It holds that $v \in P_{\zeta,\eta}$ is a solution of* (1.1) *if and only if $Tv = v$.*

Proof. If $v \in P_{\zeta,\eta}$ and $Tv = v$, then we have

$$(-1)^n v^{\Delta^{2n}}(t) = (-1)^n Tv^{\Delta^{2n}}(t) = w(t)f(t,v(t-c)), \tag{3.20}$$

and for any $0 \le i \le n-1$,

$$v^{\Delta^{2i}}(a) - \beta_{i+1}v^{\Delta^{2i+1}}(a) = \alpha_{i+1}v^{\Delta^{2i}}(\varpi), \qquad \gamma_{i+1}v^{\Delta^{2i}}(\varpi) = v^{\Delta^{2i}}(b). \tag{3.21}$$

From [8, Lemma 3.1], we know that $v(t) \ge 0$ on $[a,b]$. So we conclude that v is the solution of BVP (1.1). □

For convenience, we list the following notations and assumptions:

$$\begin{aligned}&R = \left(\mu K\int_p^q \|G_n(\cdot,s)\|w(s)\Delta s\right)^{-1}, \qquad \mu = \min\left\{\frac{h_1(p)L}{K}, \frac{h_1(q)L}{K}\right\}; \\ &\kappa = \left[K\int_a^b \|G_n(\cdot,s)\|w(s)\Delta s\right]^{-1};\end{aligned} \tag{3.22}$$

$$f_{\mu\xi}^{\xi} := \frac{f(t,u(t-c))}{u(t)}, \quad t \in [p,q], \ u \in [\mu\xi, \xi]; \tag{3.23}$$

$$f_{\rho}^{\zeta} := \frac{f(t,u(t-c))}{u(t)}, \quad t \in \Upsilon_2, \ u \in [\rho, \zeta]; \tag{3.24}$$

$$S(\rho) = \sup_{u \in \partial P_\rho} K \int_{\Upsilon_2} \|G_n(\cdot, s)\| w(s) f(s, u(s-c)) \Delta s, \quad \rho > 0. \tag{3.25}$$

From condition (C2) and (3.12), we have $S(\rho) < +\infty$.

Theorem 3.3. *Assume that there exist positive constants ρ, ζ, ξ, r with $\zeta < \mu\xi$, $r < \kappa$ and $\zeta \geq \kappa S(\rho)/(\kappa - r)$ such that*

(i) $f_{\mu\xi}^{\xi} > R$ *and* $f_{\rho}^{\zeta} < r$;

(ii) $f(t, \varphi(t-c))/u(t) < r$, *for all* $t \in \Upsilon_1$ *and* $u \in [\rho, \zeta]$.

If (C1), (C2), and (C3) hold, then the boundary value problem (1.1) *has at least one positive solution* $\widetilde{u}$ *such that*

$$\widetilde{u}(t) = \begin{cases} \varphi(t), & \text{if } t \in [a-c, a), \\ u^*(t), & \text{if } t \in [a,b], \end{cases} \qquad \zeta \leq \|u^*\| \leq \xi. \tag{3.26}$$

Proof. Define the operator $T : P_{\zeta,\xi} \to P$ by (3.6). From (i) and (3.23), it follows that there exists $\varepsilon_1 > 0$ such that

$$f(t, u(t-c)) \geq (R + \varepsilon_1) u(t), \quad \text{for } t \in [p,q], \ u \in [\mu\xi, \xi]. \tag{3.27}$$

We claim that

$$Tu \nleq u, \quad \forall u \in \partial P_\xi. \tag{3.28}$$

If it is false, then there exists some $u_1 \in \partial P_\xi$ with $Tu_1 \leq u_1$, that is, $u_1 - Tu_1 \in P$ which implies that $u_1(t) \geq Tu_1(t)$ for $t \in [a,b]$.

Set

$$\lambda = \min\{u_1(t) : t \in [p,q]\} \geq \min\left\{\frac{h_1(p)L}{K}, \frac{h_1(q)L}{K}\right\} \|u_1\| = \mu\xi. \tag{3.29}$$

We know from (2.22) and (3.27) that for $t \in [p, q]$,

$$\begin{aligned}
u_1(t) &\geq Tu_1(t) \\
&= \int_a^b H_n(t,s)w(s)f(s,u_1(s-c))\Delta s \\
&= \int_{Y_1} H_n(t,s)w(s)f(s,u_1(s-c))\Delta s + \int_{Y_2} H_n(t,s)w(s)f(s,u_1(s-c))\Delta s \\
&\geq \int_p^q H_n(t,s)w(s)f(s,u_1(s-c))\Delta s \\
&\geq \min\{h_1(p), h_1(q)\} L \int_p^q \|G_n(\cdot,s)\| w(s) f(s,u_1(s-c))\Delta s \\
&\geq (R+\varepsilon_1) \min_{t\in[p,q]} u_1(t) \mu K \int_p^q \|G_n(\cdot,s)\| w(s)\Delta s \\
&\geq \lambda R\left[\mu K \int_p^q \|G_n(\cdot,s)\| w(s)\Delta s\right] + \lambda\varepsilon_1 \mu K \int_p^q \|G_n(\cdot,s)\| w(s)\Delta s \\
&= \lambda + \lambda\varepsilon_1 \mu K \int_p^q \|G_n(\cdot,s)\| w(s)\Delta s,
\end{aligned} \tag{3.30}$$

the first inequality of (C2) implies that

$$u_1(t) > \lambda, \quad \forall t \in [p,q]. \tag{3.31}$$

Clearly, (3.31) contradicts (3.29). This means that (3.28) holds.

Next we will show that

$$Tu \not\geq u, \quad \forall u \in P_\zeta. \tag{3.32}$$

Suppose on the contrary that there exists some $u_2 \in \partial P_\zeta$ with $u_2 \leq Tu_2$ for all $t \in [a,b]$.

For $(t,u) \in Y_2 \times [\rho, \zeta]$, from (i) and (3.24), there exists $\varepsilon_2 > 0$ such that

$$f(t,u(t-c)) \leq (r-\varepsilon_2)u(t). \tag{3.33}$$

and for $(t,u) \in Y_1 \times [\rho,\zeta]$, there exists $\varepsilon_2 > 0$, from (ii), such that

$$f(t,\varphi(t-c)) \leq (r-\varepsilon_2)u(t). \tag{3.34}$$

Put

$$Y_3 := \{t \in Y_2 : u_2(t) > \rho\}, \qquad \widetilde{u}_2(t) = \begin{cases} \min\{u_2(t),\rho\}, & t \in Y_2, \\ \rho, & t \in Y_1. \end{cases} \tag{3.35}$$

If $Y_3 = \emptyset$, then we take $\tilde{u}_2(t) = \rho$. It is easy to see that $(h_1(t-c)L\zeta)/K \le u_2(t-c) \le \|u_2\| = \zeta$ for $t \in Y_2$ and $\tilde{u}_2(t) \in C^+[a,b]$, $\|\tilde{u}_2\| = \rho$, that is, $\tilde{u}_2 \in \partial P_\rho$. From (3.33) and (3.34), we find that

$$\begin{aligned}
\|Tu_2\| &= \sup_{t\in[a,b]}\int_a^b H_n(t,s)w(s)f(s,u_2(s-c))\Delta s \\
&\le K\int_a^b \|G_n(\cdot,s)\|w(s)f(s,u_2(s-c))\Delta s \\
&= K\int_{Y_1}\|G_n(\cdot,s)\|w(s)f(s,\varphi(s-c))\Delta s + K\int_{Y_3}\|G_n(\cdot,s)\|w(s)f(s,u_2(s-c))\Delta s \\
&\quad + K\int_{Y_2\setminus Y_3}\|G_n(\cdot,s)\|w(s)f(s,u_2(s-c))\Delta s \\
&\le (r-\varepsilon_2)\max_{t\in Y_1} u_2(t)\int_{Y_1}\|G_n(\cdot,s)\|w(s)\Delta s \\
&\quad + \sup_{(t,u_2)\in Y_3\times[\rho,\zeta]} f(t,u_2(t-c))K\int_{Y_3}\|G_n(\cdot,s)\|w(s)\Delta s \\
&\quad + \sup_{\tilde{u}_2\in\partial P_\rho} K\int_{Y_2}\|G_n(\cdot,s)\|w(s)f(s,\tilde{u}_2(s-c))\Delta s \\
&\le \zeta r K\int_a^b \|G_n(\cdot,s)\|w(s)\Delta s + S(\rho) - \zeta\varepsilon_2 K\int_a^b\|G_n(\cdot,s)\|w(s)\Delta s \\
&= \zeta r\kappa^{-1} - \zeta\varepsilon_2\kappa^{-1} + S(\rho) \\
&< \zeta = \|u_2\|
\end{aligned} \tag{3.36}$$

yielding a contradiction with $u_2 \le Tu_2$ for all $t \in [a,b]$. This means that (3.32) holds. Therefore, from (3.28), (3.32) and Lemma 2.9, we conclude that the operator T has at least one fixed point $u^* \in P_{\zeta,\xi}$. From the definition of the cone P and (2.18), we see that $u^*(t) > 0$ for all $t \in (a,b)$. Thus, Proposition 3.2 implies that u^* is a solution of BVP (1.1). So we obtain the desired result. □

Adopting the same argument as in Theorem 3.3 , we obtain the following results.

Corollary 3.4. *Let $\rho, \zeta, r, f_\rho^\zeta$ be as in Theorem 3.3. Suppose that (ii) of Theorem 3.3 holds and $\lim_{\xi\to\infty} f_{\mu\xi}^{\xi} = +\infty$. If (C1), (C2), and (C3) holds , then boundary value problem* (1.1) *has at least one positive solution $\hat{u} \in P_{\zeta,\eta}$ such that*

$$\hat{u}(t) = \begin{cases} \varphi(t), & \text{if } t \in [a-c,a), \\ u^{**}(t), & \text{if } t \in [a,b], \end{cases} \tag{3.37}$$
$$\zeta \le \|u^{**}\| \le \eta, \quad \zeta < \mu\eta.$$

Theorem 3.5. *Assume that there exist positive constants $\rho_i, \zeta_i, \xi_i, r$ with $\zeta_i < \mu\xi_i$, $r < \kappa$ and $\zeta_i \geq \kappa S(\rho_i)/(\kappa - r)$, $i = 1, 2, \ldots, m$ such that*

(iii) $f_{\mu\xi_i}^{\xi_i} > R$ *and* $f_{\rho_i}^{\zeta_i} < r$;

(iv) $f(t, \varphi(t-c))/u(t) < r$, *for all* $t \in \Upsilon_1$ *and* $u \in [\rho_i, \zeta_i]$.

If (C1), (C2), and (C3) hold, then boundary value problem (1.1) *has at least* m *positive solutions* $\widetilde{u}_i \in P_{\zeta_i, \xi_i}$ *such that for* $i = 1, 2, \ldots, m$

$$\widetilde{u}_i(t) = \begin{cases} \varphi(t), & \text{if } t \in [a-c, a), \\ u_i^*(t), & \text{if } t \in [a, b] \end{cases} \qquad \zeta \leq \|u_i^*\| \leq \xi. \tag{3.38}$$

Example 3.6. Let $\mathbb{T} = \mathbb{R}$. Consider the following singular three-point boundary value problems for delay four-order dynamic equations:

$$\begin{gathered} u^{(4)}(t) + f(t, u(t-1)) = 0, \quad t \in [0, 4], \\ u(0) = \frac{1}{2}u(1), \qquad 2u(1) = u(4), \\ u''(0) = \frac{1}{2}u''(1), \qquad 2u''(1) = u''(4), \\ u(t) = e^t, \quad t \in [-1, 0), \end{gathered} \tag{3.39}$$

where, for any $t \in [0, 4]$, $\rho = 1$, $\zeta = 1480$, $\mu = 0.112$, $\xi = 13500$, $M_1 = 1$ and $M_2 = 1/50^2$,

$$f(t, u(t-1)) = \begin{cases} 2M_1 u(t), & (t, u) \in (1, 4] \times [\xi, +\infty), \\ M_1 u(t)\left(1 + \sin\dfrac{\pi(u(t) - \vartheta\phi)}{2(\vartheta - \vartheta\phi)} + \cos\dfrac{\pi(u(t) - \vartheta\phi)}{2(\vartheta - \vartheta\phi)}\right), & (t, u) \in (1, 4] \times [\mu\xi, \xi], \\ \dfrac{1}{2}M_2 u(t)\cos\dfrac{\pi(u(t) - \mu)}{2(\vartheta\phi - \mu)} + 2M_1\vartheta\phi\sin\dfrac{\pi(u(t) - \mu)}{2(\vartheta\phi - \mu)}, & (t, u) \in (1, 4] \times [\zeta, \mu\xi], \\ \dfrac{1}{2}M_2 u(t)\left[2 - \sin\dfrac{\pi(u(t) - \varrho)}{2(\mu - \varrho)} - \cos\dfrac{\pi(u(t) - \varrho)}{2(\mu - \varrho)}\right], & (t, u) \in (1, 4] \times [\rho, \zeta], \\ \rho u(t)^{-1/2} - u(t)^{1/2} + \dfrac{1}{2}M_2\rho, & (t, u) \in (1, 4] \times (0, \rho], \\ \dfrac{1}{2}M_2 u(t), & (t, u) \in [0, 1) \times \mathbb{R}. \end{cases} \tag{3.40}$$

Clearly, we know that

$$\begin{gathered}
\alpha = \frac{1}{2}, \qquad \beta = 0, \qquad \gamma = 2, \qquad \eta = 1, \qquad \delta = \frac{5}{4}, \qquad d = \frac{1}{2}, \\
p = \frac{3}{2}, \quad q = \frac{7}{2}, \quad h_i(t) = \min\left\{\frac{t}{5}, \frac{4-t}{8}\right\}, \quad i = 1, 2, \\
G(4, s) = 12s \quad (s \in [0, 1]), \qquad G(4, s) = 4(4-s) \quad (s \in [1, 3]), \\
G(s, s) = (4-s)(1+s) \quad (s \in [3, 4]).
\end{gathered} \tag{3.41}$$

Simple computations yield

$$\begin{aligned}
K &= \int_0^4 \|G_1(\cdot, s)\| ds = \int_0^1 12s ds + \int_1^3 4(4-s) ds + \int_3^4 (1+s)(4-s) ds = 24.17, \\
L &= \int_0^4 \|G_1(\cdot, s) h_2(s) ds \\
&= \int_0^1 12s \frac{s}{5} ds + \int_1^{20/13} 4(4-s) \frac{s}{5} ds + \int_{20/13}^3 4(4-s) \frac{4-s}{8} ds + \int_3^4 \frac{(4-s)^2(1+s)}{8} ds \\
&= 4.695, \\
\mu &= \min\left\{\frac{h_1(p)L}{K}, \frac{h_1(q)L}{K}\right\} = 0.112, \\
R &= \left(\mu K \int_{3/2}^{7/2} \|G_2(\cdot, s)\| ds\right)^{-1} = 0.282, \\
\kappa &= \left[K \int_0^4 \|G_2(\cdot, s)\| ds\right]^{-1} = \frac{1}{24.17^2}.
\end{aligned} \tag{3.42}$$

Obviously,

$$\lim_{m\to\infty} \sup_{u \in P_{\zeta,\eta}} K \int_{\Upsilon_m} \|G_2(\cdot, s)\| \ f(s, u(s-c)) ds = 0, \quad \forall\, 0 < \zeta < \eta. \tag{3.43}$$

If $(t, u) \in (1, 4] \times (0, 1]$, then we have

$$h(t) = h(t)\rho \le u(t) \le \rho = 1. \tag{3.44}$$

Therefore, we get

$$f(t, u(t-1)) \le h(t)^{-1/2} - h(t)^{1/2} + \frac{1}{2} M_2, \quad \text{for } (t, u) \in (1, 4] \times (0, 1]. \tag{3.45}$$

From (3.25), it follows that

$$\begin{aligned}
S(1) &= \sup_{u\in\partial P_1} K\int_1^4 \|G_2(\cdot,s)\| f(s,u(s-1))ds \\
&\le K\int_1^{20/13} 12s\left(\left(\frac{5}{s}\right)^{1/2} - \left(\frac{s}{5}\right)^{1/2} + \frac{1}{2}M_2\right)ds \\
&\quad + K\int_{20/13}^{3} 4(4-s)\left(\left(\frac{8}{4-s}\right)^{1/2} - \left(\frac{4-s}{8}\right)^{1/2} + \frac{1}{2}M_2\right)ds \\
&\quad + K\int_{3}^{4} (1+s)(4-s)\left(\left(\frac{8}{4-s}\right)^{1/2} - \left(\frac{4-s}{8}\right)^{1/2} + \frac{1}{2}M_2\right)ds \\
&\le 1120.
\end{aligned} \tag{3.46}$$

Thus,

$$\zeta = 1480 \ge \frac{\kappa S(1)}{\kappa - r} \approx 1461.37, \quad \xi = 13500, \quad \zeta < \mu\xi. \tag{3.47}$$

Therefore, by Theorem 3.3 , the BVP (3.39) has at least one positive solution $\widetilde{u}$ such that

$$\begin{gathered}
\widetilde{u}(t) = \begin{cases} e^t, & \text{if } t \in [-1,0), \\ u^*(t), & \text{if } t \in [0,4], \end{cases} \\
1480 \le \|u^*\| \le 13500.
\end{gathered} \tag{3.48}$$

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